

ESTIMATES FOR DIRICHLET EIGENFUNCTIONS

M. VAN DEN BERG AND E. BOLTHAUSEN

ABSTRACT

Estimates for the Dirichlet eigenfunctions near the boundary of an open, bounded set in euclidean space are obtained. It is assumed that the boundary satisfies a uniform capacitary density condition.

1. Introduction

Let D be an open, bounded set in euclidean space \mathbb{R}^m ($m = 2, 3, \dots$) with boundary ∂D . Let $-\Delta_D$ be the Dirichlet laplacian for D . The spectrum of $-\Delta_D$ is discrete and consists of eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots$ with a corresponding orthonormal set of eigenfunctions $\{\phi_1, \phi_2, \dots\}$. The behaviour of the eigenfunctions near the boundary ∂D of D has been investigated by several authors under a variety of assumptions on the geometry of D [1–8, 11–16].

In this paper we obtain pointwise bounds on the eigenfunctions under the assumption that ∂D satisfies a uniform capacitary density condition. Denote by $\text{Cap}(A)$ the newtonian capacity of a compact set $A \subset \mathbb{R}^m$ ($m = 3, 4, \dots$) or the logarithmic capacity of a compact set $A \subset \mathbb{R}^2$. For $x \in \mathbb{R}^m$ and $r > 0$ we define

$$B(x; r) = \{y \in \mathbb{R}^m : |y - x| \leq r\}, \quad (1.1)$$

and for a non-empty set $G \subset \mathbb{R}^m$

$$\text{diam}(G) = \sup\{|x_1 - x_2| : x_1 \in G, x_2 \in G\}. \quad (1.2)$$

DEFINITION 1.1. Let $D \subset \mathbb{R}^m$ ($m = 2, 3, \dots$) be an open set with boundary ∂D . Then ∂D satisfies an α -uniform capacitary density condition if for some $\alpha \in (0, 1]$

$$\text{Cap}((\partial D) \cap B(x; r)) \geq \alpha \text{Cap}(B(x; r)), \quad x \in \partial D, \quad 0 < r < \text{diam}(D). \quad (1.3)$$

Condition (1.3) guarantees that all points of ∂D are regular, and in particular that $\lim_{x \rightarrow x_0} \phi_j(x) = 0$ for all $x_0 \in \partial D$. Definition 1.1 has been introduced in [10] in a study of the partition function of the Dirichlet laplacian on open sets with a non-smooth or fractal boundary.

Let $d: D \rightarrow (0, \infty)$ denote the distance function

$$d(x) = \min\{|x - y| : y \in \partial D\}, \quad (1.4)$$

and let R be the inradius of D , defined by

$$R = \max_{x \in D} d(x). \quad (1.5)$$

The main results of this paper are the following.

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THEOREM 1.2. *Let D be an open, bounded set in \mathbb{R}^2 . Suppose ∂D satisfies (1.3) for some $\alpha > 0$. Then for $j = 1, 2, \dots$ and all $x \in D$ such that $d(x) < \lambda_j^{-1/2}$*

$$|\phi_j(x)| \leq \left\{ 6\lambda_j \log(2/\alpha^{2\pi}) \frac{-1}{\log(d(x)\lambda_j^{1/2})} \right\}^{1/2}. \quad (1.6)$$

THEOREM 1.3. *Let D be an open, bounded set in \mathbb{R}^m ($m = 3, 4, \dots$). Suppose ∂D satisfies (1.3) for some $\alpha > 0$. Let $j = 1, 2, \dots$ and suppose $x \in D$ satisfies*

$$d(x)\lambda_j^{1/2} \leq \left(\frac{\alpha^6}{2^{13}} \right)^{1+\gamma(m-1)/(m-2)}. \quad (1.7)$$

Then

$$|\phi_j(x)| \leq 2\lambda_j^{m/4} (d(x)\lambda_j^{1/2})^{(1/2)((1/\gamma)+(m-1)/(m-2))^{-1}}, \quad (1.8)$$

where

$$\gamma = \frac{3^{-m-1}\alpha}{\log(2(2/\alpha)^{1/(m-2)})}. \quad (1.9)$$

The bounds in (1.6) and (1.8) are being complemented by the following well-known estimate [11, Lemma 3.1].

THEOREM 1.4. *Let D be an open, bounded set in \mathbb{R}^m ($m = 2, 3, \dots$). Then for $j = 1, 2, \dots$ and $x \in D$*

$$|\phi_j(x)| \leq \lambda_j^{m/4}. \quad (1.10)$$

The bounds of Theorems 1.2 and 1.3 are in general not sharp. For example if D is open, bounded and ∂D is smooth then the eigenfunctions are comparable with $d(x)$. If D is open, bounded and simply connected in \mathbb{R}^2 then it was shown by Bañuelos [3, Corollary 2.3b] that ϕ_1 is comparable to the hyperbolic distance induced by the conformal map F from the unit disc onto D . By Koebe's 1/4 theorem [17] one then obtains that

$$\phi_1(x) \leq Cd(x)^{1/2} \quad (1.11)$$

for some constant C depending on D . We will use the ideas of [3] to prove (in Section 5) the following refinement of (1.11).

THEOREM 1.5. *Let D be an open, simply connected set in \mathbb{R}^2 with volume $|D|$. Then for $j = 1, 2, \dots$*

$$|\phi_j(x)| \leq 2^{9/2}\pi^{1/4}j|D|^{1/4}R^{-2}d(x)^{1/2}. \quad (1.12)$$

The following example (see [13, 4.6.7]) shows that Theorem 1.5 is sharp.

EXAMPLE 1.6. Let $U \subset \mathbb{R}^m$ be the conical region in polar coordinates defined by

$$U = \{(r, \omega) : 0 < r < 1, \omega \in \Omega\}, \quad (1.13)$$

where Ω is an open subset of the unit sphere S^{m-1} . Then

$$\phi_1(r, \omega) \asymp r^{\beta(\Omega)}, \quad (1.14)$$

where $\beta(\Omega)$ is the positive solution of

$$\beta(\beta + m - 2) = \lambda_1(\Omega), \quad (1.15)$$

and where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplace–Beltrami operator on Ω with Dirichlet conditions of $\partial\Omega$. In particular if $m = 2$ and

$$\Omega_0 = \{\omega \in S^1 : 0 < \omega < 2\pi\}, \quad (1.16)$$

then

$$\lambda_1(\Omega_0) = \frac{1}{4} \quad (1.17)$$

and by (1.15)

$$\beta(\Omega_0) = \frac{1}{2}, \quad (1.18)$$

which shows that the exponent in (1.12) cannot be improved.

The main idea of the proofs of Theorems 1.2 and 1.3 goes back to Brossard and Carmona [10] who obtained estimates for the Dirichlet heat kernel $p_D(x, x; t)$ for x near D . We improve their lemma [10, Lemma 3.5] and its proof (see also [9]). In the proof of Theorem 1.3 we also require a refinement of Wiener's test [18].

Let $(B(s), s \geq 0; \mathbb{P}_x, x \in \mathbb{R}^m)$ be a brownian motion associated to $-\Delta + \partial/\partial t$. Let T_D denote the first exit time of the brownian motion from D :

$$T_D = \inf\{s \geq 0 : B(s) \in \partial D\}. \quad (1.19)$$

For a compact set K we also define the first entry time

$$\tau_K = \inf\{s > 0 : B(s) \in K\}. \quad (1.20)$$

Then

$$\mathbb{P}_x[T_D > t] = \int_D p_D(x, y; t) dy. \quad (1.21)$$

By the eigenfunction expansion of the heat kernel and by the semigroup property we have

$$e^{-t\lambda_j} \phi_j^2(x) \leq \sum_{j=1}^{\infty} e^{-t\lambda_j} \phi_j^2(x) = p_D(x, x; t) = \int_D p_D^2(x, y; t/2) dy. \quad (1.22)$$

Since the Dirichlet heat kernel is monotone in D

$$p_D(x, y; t/2) \leq p_{\mathbb{R}^m}(x, y; t/2) \leq (2\pi t)^{-m/2}. \quad (1.23)$$

Hence

$$e^{-t\lambda_j} \phi_j^2(x) \leq (2\pi t)^{-m/2} \int_D p_D(x, y; t/2) dy = (2\pi t)^{-m/2} \mathbb{P}_x[T_D > t/2]. \quad (1.24)$$

The choice

$$t = 2\lambda_j^{-1} \quad (1.25)$$

yields for $m = 2, 3, \dots$

$$|\phi_j(x)| \leq e(4\pi)^{-m/4} \lambda_j^{m/4} (\mathbb{P}_x[T_D > \lambda_j^{-1}])^{1/2} \leq \lambda_j^{m/4} (\mathbb{P}_x[T_D > \lambda_j^{-1}])^{1/2}. \quad (1.26)$$

This proves Theorem 1.4 since $\mathbb{P}_x[T_D > \lambda_j^{-1}] \leq 1$.

In Sections 2 and 3 we obtain the upper bounds for $\mathbb{P}_x[T_D > \lambda_j^{-1}]$ in the cases $m = 2$ and $m = 3, 4, \dots$ respectively. In the proof of Theorem 1.3 we use the following modification of Wiener's test. See also [20, Theorem 4.7, p. 73] for related refinements of Wiener's test.

THEOREM 1.7. *Let D be an open, bounded set in \mathbb{R}^m ($m = 3, 4, \dots$). Suppose ∂D satisfies (1.3) for some $\alpha > 0$. If $x \in D$ satisfies $d(x) \leq (\alpha^6/2^{13}) \text{diam}(D)$, then for any*

$$a \in \left[\frac{2^{13}}{\alpha^6}, \text{diam}(D)/d(x) \right] \quad (1.27)$$

one has

$$\mathbb{P}_x[\tau_{(\partial D) \cap B(x; ad(x))} < \infty] \geq 1 - 2a^{-\gamma}, \quad (1.28)$$

where γ is given by (1.9).

The proof of Theorem 1.7 will be deferred to Section 4.

2. Proof of Theorem 1.2

Let $m = 2$ and define the Green function

$$g(x, y) = -(2\pi)^{-1} \log |x - y|. \quad (2.1)$$

The equilibrium measure on a compact set $K \subset \mathbb{R}^2$ is the unique probability measure μ_K concentrated on K for which

$$u_K(x) = \int_K g(x, y) \mu_K(dy) \quad (2.2)$$

is constant on the regular points of K . The function u_K is the equilibrium potential of K and its value $R(K)$ on the regular points of K is the Robin constant. The logarithmic capacity is defined by

$$\text{Cap}(K) = e^{-R(K)}. \quad (2.3)$$

Then

$$\text{Cap}(K) = \exp - \left\{ \inf_{\mu \in P(K)} \int_K \int_K g(x, y) \mu(dx) \mu(dy) \right\}, \quad (2.4)$$

where $P(K)$ is the set of all probability measures supported by K . Moreover

$$\text{Cap}(B(x; r)) = r^{1/(2\pi)} \quad (2.5)$$

[18, Chapter 3, Proposition 4.11]. Define

$$B^\circ(x; r) = \{y \in \mathbb{R}^m : |y - x| < r\}. \quad (2.6)$$

Let $a > 4$. Then

$$\begin{aligned} \mathbb{P}_x[T_D > t] &\leq \mathbb{P}_x[\tau_{(\partial D) \cap B(x; 2d(x))} > t] \\ &= \mathbb{P}_x[\tau_{(\partial D) \cap B(x; 2d(x))} > t, \tau_{(\partial D) \cap B(x; 2d(x))} > T_{B^\circ(x; ad(x))}] \\ &\quad + \mathbb{P}_x[\tau_{(\partial D) \cap B(x; 2d(x))} > t, \tau_{(\partial D) \cap B(x; 2d(x))} \leq T_{B^\circ(x; ad(x))}] \\ &\leq 1 - \mathbb{P}_x[\tau_{(\partial D) \cap B(x; 2d(x))} \leq T_{B^\circ(x; ad(x))}] + \mathbb{P}_x[T_{B^\circ(x; ad(x))} > t]. \end{aligned} \quad (2.7)$$

Let H be an open half space in \mathbb{R}^2 containing $B^\circ(x; ad(x))$ such that ∂H is tangent to $\partial B^\circ(x; ad(x))$. Then

$$\begin{aligned} \mathbb{P}_x[T_{B^\circ(x; ad(x))} > t] &\leq \mathbb{P}_x[T_H > t] \\ &= (\pi t)^{-1/2} \int_{[0, ad(x))} e^{-q^2/(4t)} dq \leq ad(x) (\pi t)^{-1/2}. \end{aligned} \quad (2.8)$$

For compact sets $K_1 \subseteq K_2$ we have by the variational formula (2.4)

$$\text{Cap}(K_1) \leq \text{Cap}(K_2). \quad (2.9)$$

Let $x_0 \in \partial D$ be such that $d(x) = |x - x_0|$. Then $B(x; 2d(x)) \supset B(x_0; d(x))$, and by (1.3), (2.5) and (2.9)

$$\begin{aligned} \text{Cap}((\partial D) \cap B(x; 2d(x))) &\geq \text{Cap}((\partial D) \cap B(x_0; d(x))) \\ &\geq \alpha \text{Cap}(B(x_0; d(x))) = \alpha(d(x))^{1/(2\pi)}. \end{aligned} \quad (2.10)$$

By (2.3) and (2.10)

$$R((\partial D) \cap B(x; 2d(x))) \leq -(2\pi)^{-1} \log d(x) - \log \alpha. \quad (2.11)$$

By (2.3) and (2.9) we have for $K_1 \subseteq K_2$

$$R(K_1) \geq R(K_2). \quad (2.12)$$

Hence

$$R((\partial D) \cap B(x; 2d(x))) \geq R(B(x; 2d(x))) \geq -(2\pi)^{-1} \log(2d(x)). \quad (2.13)$$

Moreover, by (2.1) and (2.2)

$$\begin{aligned} & \sup\{u_{(\partial D) \cap B(x; 2d(x))}(y) : y \in \partial B(x; ad(x))\} \\ & \leq -(2\pi)^{-1} \log((a-2)d(x)) \int_{(\partial D) \cap B(x; 2d(x))} \mu_{(\partial D) \cap B(x; 2d(x))}(dy) \\ & = -(2\pi)^{-1} \log((a-2)d(x)). \end{aligned} \quad (2.14)$$

Following the proof of [11, Lemma 3.5] we define for $r > 0$

$$m(r) = -(2\pi)^{-1} \log((a-2)r), \quad (2.15)$$

and $h: \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$h(y) = (R((\partial D) \cap B(x; 2d(x))) - m(d(x)))^{-1} (u_{(\partial D) \cap B(x; 2d(x))}(y) - m(d(x))). \quad (2.16)$$

Now h is superharmonic, harmonic outside $(\partial D) \cap B(x; 2d(x))$, equal to one on $(\partial D) \cap B(x; 2d(x))$, and by (2.14) negative on $\partial B(x; ad(x))$. Hence

$$\mathbb{P}_x[\tau_{(\partial D) \cap B(x; 2d(x))} \leq T_{B^c(x; ad(x))}] \geq h(x) = \frac{u_{(\partial D) \cap B(x; 2d(x))}(x) - m(d(x))}{R((\partial D) \cap B(x; 2d(x))) - m(d(x))}. \quad (2.17)$$

But

$$u_{(\partial D) \cap B(x; 2d(x))}(x) \geq -\frac{1}{2\pi} \log(2d(x)), \quad (2.18)$$

so that by (2.11), (2.15), (2.17) and (2.18)

$$\mathbb{P}_x[\tau_{(\partial D) \cap B(x; 2d(x))} \leq T_{B^c(x; ad(x))}] \geq \frac{\log(a-2) - \log 2}{\log(a-2) - \log \alpha^{2\pi}}. \quad (2.19)$$

Hence by (2.7), (2.8) and (2.19)

$$\mathbb{P}_x[T_D > \lambda_j^{-1}] \leq \frac{\log(2/\alpha^{2\pi})}{\log((a-2)/\alpha^{2\pi})} + \pi^{-1/2} ad(x) \lambda_j^{1/2}. \quad (2.20)$$

We make the following choice for a :

$$a-2 = \frac{2}{\lambda_j^{1/2} d(x)} \left(1 + \log \left(\frac{1}{\lambda_j^{1/2} d(x)} \right) \right)^{-1}. \quad (2.21)$$

Let $z = \lambda_j^{-1/2} d(x)^{-1}$. Then $d(x) \leq \lambda_j^{-1/2}$ implies $z \geq 1$, and $z \geq 1 + \log(z)$ implies $a \geq 4$. By (2.20) and (2.21)

$$\begin{aligned} \mathbb{P}_x[T_D > \lambda_j^{-1}] & \leq \left(\log \frac{2}{\alpha^{2\pi}} \right) \left(\log \frac{1}{\alpha^{2\pi}} + \log(2z) - \log(1 + \log z) \right)^{-1} \\ & \quad + 2\pi^{-1/2} z^{-1} + 2\pi^{-1/2} (1 + \log z)^{-1}. \end{aligned} \quad (2.22)$$

LEMMA 2.1. For $z \geq 1$

$$\log(2z) - \log(1 + \log z) \geq \frac{1}{2} \log z. \quad (2.23)$$

Proof. Inequality (2.23) is equivalent to

$$4z \geq (1 + \log z)^2. \quad (2.24)$$

But (2.24) holds for $z = 1$. Moreover for $z \geq 1$

$$4 \geq 2(1 + \log z)/z. \quad (2.25)$$

Integration of (2.25) over $[1, z]$ yields (2.24) and hence (2.23).

For $z \geq 1$, $1/z \leq (\log 1/z)^{-1}$. Hence by Lemma 2.1 and (2.22)

$$\begin{aligned} \mathbb{P}_x[T_D > \lambda_j^{-1}] &\leq \left(2 \log\left(\frac{2}{\alpha^{2\pi}}\right) + \frac{4}{\pi^{1/2}}\right)(\log z)^{-1} \\ &\leq 6\left(\log\left(\frac{2}{\alpha^{2\pi}}\right)\right)(\log z)^{-1}. \end{aligned} \quad (2.26)$$

Theorem 1.2 follows from (1.26), (2.26) and by the definition of z . \square

3. Proof of Theorem 1.3

We define for $m = 3, 4, \dots$ the Green function on \mathbb{R}^m by

$$g(x, y) = \frac{1}{c(m)} |x - y|^{2-m}, \quad (3.1)$$

where

$$c(m) = 4\pi^{m/2}(\Gamma((m-2)/2))^{-1}. \quad (3.2)$$

For a compact set $K \subset \mathbb{R}^m$ the equilibrium measure μ_K is the unique non-negative measure on K satisfying

$$\mathbb{P}_x[\tau_K < \infty] = \int g(x, y) \mu_K(dy). \quad (3.3)$$

The newtonian capacity of K is defined by

$$\text{Cap}(K) = \mu_K(K). \quad (3.4)$$

The capacity of a ball is

$$\text{Cap}(B(0; r)) = c(m) r^{m-2}. \quad (3.5)$$

Again, there is a variational description

$$\text{Cap}(K) = \left\{ \inf_{\mu \in P(K)} \int \int g(x, y) \mu(dx) \mu(dy) \right\}^{-1}, \quad (3.6)$$

where $P(K)$ is the set of all probability measures supported by K . If K_1, K_2 are compact sets with $K_1 \subseteq K_2$ then $\text{Cap}(K_1) \leq \text{Cap}(K_2)$. For these facts, see for example [18, Chapter 3].

To prove Theorem 1.3 we adapt [10, Lemma 3.5]. Let $b > a > 1$. Then by the strong Markov property

$$\begin{aligned} \mathbb{P}_x[T_D > t] &\leq 1 - \mathbb{P}_x[\tau_{(\partial D) \cap B(x; ad(x))} \leq t] \\ &\leq 1 - \mathbb{P}_x[\tau_{(\partial D) \cap B(x; ad(x))} \leq T_{B^o(x; bd(x))}] + \mathbb{P}_x[T_{B^o(x; bd(x))} > t] \\ &\leq 1 - \mathbb{P}_x[\tau_{(\partial D) \cap B(x; ad(x))} < \infty] \\ &\quad + \mathbb{E}_x[\mathbb{P}_{B(T_{B^o(x; bd(x))})}[\tau_{(\partial D) \cap B(x; ad(x))} < \infty]] + \mathbb{P}_x[T_{B^o(x; bd(x))} > t]. \end{aligned} \quad (3.7)$$

To estimate the third term in the right-hand side of (3.7) we let y be such that $|y-x| = bd(x)$. Then

$$\mathbb{P}_y[\tau_{(\partial D) \cap B(x; ad(x))}] \leq \mathbb{P}_y[\tau_{B(x; ad(x))} < \infty] = \left(\frac{a}{b}\right)^{m-2}. \quad (3.8)$$

The fourth term in (3.7) is again estimated by (2.8). Hence

$$\mathbb{P}_x[T_D > t] \leq 1 - \mathbb{P}_x[\tau_{(\partial D) \cap B(x; ad(x))} < \infty] + \left(\frac{a}{b}\right)^{m-2} + bd(x)(\pi t)^{-1/2}. \quad (3.9)$$

Choose a and b as follows:

$$a = Ad(x)^{-\beta_1}, \quad (3.10)$$

$$b = B(d(x))^{\beta_2-1}, \quad (3.11)$$

where β_1, β_2, A and B are the solutions of

$$\beta_1 \gamma = (1 - \beta_1 - \beta_2)(m-2) = \beta_2, \quad (3.12)$$

$$A^{-\gamma} = \left(\frac{A}{B}\right)^{m-2} = B\lambda_j^{1/2}. \quad (3.13)$$

From (3.12) we obtain

$$\beta_2 = \beta_1 \gamma = \left(\frac{1}{\gamma} + \frac{m-1}{m-2}\right)^{-1}, \quad (3.14)$$

and from (3.13) we obtain

$$B\lambda_j^{1/2} = A^{-\gamma} = \lambda_j^{(1/2)((1/\gamma)+(m-1)/(m-2))^{-1}}. \quad (3.15)$$

If we can show that (1.7) implies (1.27) and the requirement $b \geq a$, then by (3.9)–(3.11) and Theorem 1.7

$$\begin{aligned} \mathbb{P}_x[T_D > \lambda_j^{-1}] &\leq 2A^{-\gamma}d(x)^{\beta_1\gamma} + \left(\frac{A}{B}\right)^{m-2} d(x)^{(1-\beta_1-\beta_2)(m-2)} \\ &\quad + Bd(x)^{\beta_2}\lambda_j^{1/2}. \end{aligned} \quad (3.16)$$

Substitution of the values of β_1, β_2, A and B respectively in (3.16) gives

$$\mathbb{P}_x[T_D > \lambda_j^{-1}] \leq 4A^{-\gamma}d(x)^{\beta_1\gamma} = 4(d(x)\lambda_j^{1/2})^{((1/\gamma)+(m-1)/(m-2))^{-1}}. \quad (3.17)$$

Estimate (1.8) follows from (1.26) and (3.17).

Note that $b \geq a$ is (by (3.10), (3.11)) equivalent to showing that

$$\frac{B}{A} \geq d(x)^{1-\beta_1-\beta_2}. \quad (3.18)$$

It follows from (3.14) that

$$0 < \beta_1 + \beta_2 = \frac{1+\gamma}{1+\gamma\frac{m-1}{m-2}} < 1. \quad (3.19)$$

Since (1.7) implies $d(x) \leq \lambda_j^{-1/2}$ it is (by (3.19)) sufficient to check that (3.18) holds for $d(x) = \lambda_j^{-1/2}$, that is,

$$\frac{B}{A} \geq \lambda_j^{-(1/2)(1-\beta_1-\beta_2)}. \quad (3.20)$$

We see by (3.15) and (3.19) that (3.20) holds with the equality sign.

It remains to check the validity of (1.27). Since D is bounded, D is contained in a hypercube of sidelength $\text{diam}(D)$. By monotonicity of the Dirichlet eigenvalues [19, Chapter XIII.15, Proposition 4(a)]

$$\lambda_j \geq \lambda_1 = \frac{m\pi^2}{(\text{diam}(D))^2} > \frac{1}{(\text{diam}(D))^2}. \quad (3.21)$$

Hence the first inequality in (1.27) is satisfied if

$$\frac{1}{\lambda_j^{1/2} d(x)} \geq a. \quad (3.22)$$

By (3.10), (3.14) and (3.15)

$$a = (\lambda_j^{1/2} d(x))^{-(1/\gamma)((1/\gamma)+(m-1)/(m-2))^{-1}}. \quad (3.23)$$

Hence (3.22) is satisfied if

$$(d(x) \lambda_j^{1/2})^{1-(1/\gamma)((1/\gamma)+(m-1)/(m-2))^{-1}} \leq 1. \quad (3.24)$$

This is indeed the case because (1.7) implies $d(x) \lambda_j^{1/2} \leq 1$ and

$$1 - \frac{1}{\gamma} \left(\frac{1}{\gamma} + \frac{m-1}{m-2} \right)^{-1} > 0. \quad (3.25)$$

The second inequality in (1.27) follows directly from (3.23) and (1.7). \square

4. Proof of Theorem 1.7

For $s > r > 0$ we define the annulus

$$B(x; r, s) = B(x; s) \setminus B^o(x; r), \quad (4.1)$$

and the set

$$\partial D_i(x) = (\partial D) \cap B(x; b^i, b^{i+1}), \quad (4.2)$$

where $b > 1$ will be specified later. Let $A_i(x)$ be the event

$$A_i(x) = \{\tau_{\partial D_i(x)} < \infty\}. \quad (4.3)$$

Define

$$N = \max\{k \in \mathbb{Z} : b^{k+1} \leq a(d(x)) d(x)\}. \quad (4.4)$$

Then for any $n \leq N$

$$\{\tau_{(\partial D) \cap B(x; a(d(x)) d(x))} < \infty\} \supset \bigcup_{i=n}^N A_i(x). \quad (4.5)$$

If $b^{i+1} < d(x)$ then $A_i(x) = \emptyset$. We will choose

$$n = \min\{k \in \mathbb{Z} : b^k \geq 2d(x)\}. \quad (4.6)$$

We choose a ‘spacing’ $s \in \mathbb{Z}^+$, $s \geq 2$ (to be specified later) and use

$$\bigcup_{i=n}^N A_i(x) \supset \bigcup_{j=0}^{\lfloor (N-n)/s \rfloor} A_{n+js}(x), \quad (4.7)$$

to obtain

$$\mathbb{P}_x[\tau_{(\partial D) \cap B(x; a(d(x)) d(x))} < \infty] \geq 1 - \mathbb{P}_x \left[\bigcap_{j=0}^{\lfloor (N-n)/s \rfloor} A_{n+js}^c(x) \right]. \quad (4.8)$$

For technical reasons we replace $A_j(x)$ by $\bar{A}_j(x)$ which are defined by

$$\bar{A}_j(x) = \{\tau_{\partial D_j(x)} \leq \tau_{B(x; b^{j+s}, \infty)}\}. \quad (4.9)$$

Note that

$$\mathbb{P}_x[\bar{A}_j(x) \setminus A_j(x)] = 0, \quad (4.10)$$

and therefore

$$\mathbb{P}_x[\tau_{(\partial D) \cap B(x; a(d(x))d(x))} < \infty] \geq 1 - \mathbb{P}_x \left[\bigcap_{j=0}^{\lfloor (N-n)/s \rfloor} \bar{A}_{n+js}^c(x) \right]. \quad (4.11)$$

Next we derive a lower bound for $\mathbb{P}_y(\bar{A}_j(x))$ for $|x-y| \leq b^j$.

LEMMA 4.1. *Let*

$$b = 2 \left(\frac{2}{\alpha} \right)^{1/(m-2)}. \quad (4.12)$$

Then for $j \geq n$ satisfying $b(b^j + d(x)) \leq 2 \operatorname{diam}(D)$ and any y satisfying $|y-x| \leq b^j$

$$\mathbb{P}_y[\bar{A}_j(x)] \geq 2 \cdot 3^{-m} \alpha. \quad (4.13)$$

Proof. Let $x_0 \in \partial D$ be such that $d(x) = |x-x_0|$. One easily checks that if $b^j \geq 2d(x)$

$$\partial D_j(x) \supset (\partial D) \cap B(x_0; r, br/2), \quad (4.14)$$

where

$$r = b^j + d(x). \quad (4.15)$$

From this we obtain, by the monotonicity and subadditivity of the newtonian capacity,

$$\begin{aligned} \operatorname{Cap}(\partial D_j(x)) &\geq \operatorname{Cap}((\partial D) \cap B(x_0; r, br/2)) \\ &\geq \operatorname{Cap}((\partial D) \cap B(x_0; br/2)) - \operatorname{Cap}(B(x_0; r)) \\ &\geq \alpha \operatorname{Cap}(B(x_0; br/2)) - \operatorname{Cap}(B(x_0; r)), \end{aligned} \quad (4.16)$$

since $br/2 \leq \operatorname{diam}(D)$ by assumption. By the choice of b and by (3.5)

$$\operatorname{Cap}(\partial D_j(x)) \geq \alpha c(m) (br/2)^{m-2} - c(m) r^{m-2} = c(m) r^{m-2} \geq c(m) b^{j(m-2)}. \quad (4.17)$$

For $z \in \partial D_j(x)$ and $|y-x| \leq b^j$ we have since $b \geq 2$

$$|y-z| \leq |z-x| + |x-y| \leq b^{j+1} + b^j \leq 3b^{j+1}/2. \quad (4.18)$$

Hence by (4.17) and (4.18)

$$\begin{aligned} \mathbb{P}_y[A_j(x)] &= \int_{\partial D_j(x)} \frac{c(m)^{-1}}{|y-z|^{m-2}} \mu_{\partial D_j(x)}(dz) \\ &\geq c(m)^{-1} (3/2)^{2-m} b^{(2-m)(j+1)} \operatorname{Cap}(\partial D_j(x)) \\ &\geq c(m)^{-1} (3/2)^{2-m} b^{(2-m)(j+1)} c(m) b^{j(m-2)} \\ &= (3b/2)^{2-m}. \end{aligned} \quad (4.19)$$

Furthermore

$$\mathbb{P}_y[\bar{A}_j(x)] \geq \mathbb{P}_y[A_j(x)] - \mathbb{P}_w[\tau_{B(x; b^{j+1})} < \infty], \quad (4.20)$$

where $|w-x| = b^{j+s}$. Hence

$$\mathbb{P}_y[\bar{A}_j(x)] \geq (3b/2)^{2-m} - b^{(m-2)(1-s)}. \quad (4.21)$$

From now on we choose $s = 3$. Then by (4.21) and (4.12)

$$\mathbb{P}_y[\bar{A}_j(x)] \geq 3^{2-m} \frac{\alpha}{2} - \alpha^2 2^{2(1-m)} \geq 2 \cdot 3^{-m} \alpha. \quad (4.22)$$

□

Let

$$\mathcal{F}_j \equiv \sigma(B(t) : t \leq \tau_{B(x; b^j, \infty)}). \quad (4.23)$$

By definition of $\bar{A}_j(x)$, we have that $\bar{A}_j(x)$ is \mathcal{F}_{j+3} -measurable. Let x be such that

$$N - n \geq 3. \quad (4.24)$$

Then for any $k \in \{1, 2, \dots, [(N-n)/3]\}$ we have

$$\begin{aligned} \mathbb{P}_x \left[\bigcap_{j=0}^k \bar{A}_{n+3j}(x) \right] &= \mathbb{E}_x \left[\mathbb{P}_x[\bar{A}_{n+3k}^c(x) \mid \mathcal{F}_{n+3k}]; \bigcap_{j=0}^{k-1} \bar{A}_{n+3j}(x) \right] \\ &= \mathbb{E}_x \left[\mathbb{P}_{B(\tau_{B(x; b^{n+3k}, \infty)})}(\bar{A}_{n+3k}^c); \bigcap_{j=0}^{k-1} \bar{A}_{n+3j}(x) \right] \\ &\leq (1 - 2 \cdot 3^{-m} \alpha) \mathbb{P}_x \left[\bigcap_{j=0}^{k-1} \bar{A}_{n+3j}(x) \right]. \end{aligned} \quad (4.25)$$

From this we finally obtain

$$\begin{aligned} \mathbb{P}_x \left[\bigcap_{j=0}^{[(N-n)/3]} \bar{A}_{n+3j}(x) \right] &\leq (1 - 2 \cdot 3^{-m} \alpha)^{[(N-n)/3]} \\ &\leq \exp - \{[(N-n)/3] 2 \cdot 3^{-m} \alpha\}. \end{aligned} \quad (4.26)$$

Since $N - n \geq 3$,

$$[(N-n)/3] \geq (N-n)/6, \quad (4.27)$$

and

$$\mathbb{P}_x \left[\bigcap_{j=0}^{[(N-n)/3]} \bar{A}_{n+3j}(x) \right] \leq \exp - \{(N-n) 3^{-m-1} \alpha\}. \quad (4.28)$$

By the choice of N and n we have

$$b^{N+2} > a(d(x)) d(x) \geq b^{N+1}, \quad (4.29)$$

$$b^{n-1} < 2d(x) \leq b^n, \quad (4.30)$$

and hence

$$b^{N-n} \geq \frac{a(d(x))}{2b^3}. \quad (4.31)$$

By (4.28) and (4.31) we obtain

$$\begin{aligned} \mathbb{P}_x \left[\bigcap_{j=0}^{[(N-n)/3]} \bar{A}_{n+3j}(x) \right] &\leq \left(\frac{a(d(x))}{2b^3} \right)^{-\alpha 3^{-m-1}/\log b} \\ &= a(d(x))^{-\gamma} e^{\alpha 3^{-m} + \alpha 3^{-m-1} (\log 2)/\log b} \\ &\leq a(d(x))^{-\gamma} e^{4\alpha 3^{-m-1}} \leq 2a(d(x))^{-\gamma}, \end{aligned} \quad (4.32)$$

by definition of γ and the fact that $b \geq 2$, $m \geq 3$ and $\alpha < 1$. Estimate (1.28) follows from (4.11) and (4.32) provided $x \in D$ is such that (i) $b(b^j + d(x)) \leq 2 \operatorname{diam}(D)$ for $j = n, \dots, N$, (ii) (4.24) holds. But (i) is satisfied if

$$b^{N+1} + 2b^{N-n+1}d(x) \leq 2 \operatorname{diam}(D), \quad (4.33)$$

since $b \geq 2$. But $b^{N+1} \leq a(d(x)) d(x)$ and $2b^{N-n+1} \leq a(d(x))$. So (4.33) and hence (i) are clearly satisfied if

$$a(d(x)) d(x) \leq \operatorname{diam}(D). \quad (4.34)$$

For (4.24) to hold we have to have $b^{N+2-(n-1)} \geq b^6$. This is the case by (4.29) and (4.30) if

$$a(d(x)) \geq 2b^6. \quad (4.35)$$

But $b \leq 4/\alpha$ since $0 < \alpha < 1$ and $m = 3, 4, \dots$. Hence (4.35) and (4.24) hold if

$$a(d(x)) \geq \frac{2^{13}}{\alpha^6}. \quad (4.36)$$

This completes the proof of Theorem 1.7. \square

5. Proof of Theorem 1.5

Let $G_D(\cdot, \cdot)$ be the Green function for $-\Delta_D$. Then

$$G_D(x, y) = \int_0^\infty p_D(x, y; t) dt, \quad (5.1)$$

and any Dirichlet eigenfunction of $-\Delta_D$ satisfies

$$\varphi_j(x) = \lambda_j \int_D G_D(x, y) \varphi_j(y) dy. \quad (5.2)$$

By the Cauchy-Schwarz inequality

$$|\varphi_j(x)| \leq \lambda_j \left\{ \int_D G_D^2(x, y) dy \right\}^{1/2} \quad (5.3)$$

since $\|\varphi_j\|_2 = 1$.

LEMMA 5.1. For $j = 1, 2, \dots$

$$\lambda_j \leq 8\pi j R^{-2}. \quad (5.4)$$

Proof. By definition of R , D contains an open ball with radius R . Hence D contains an open square with sidelength $R\sqrt{2}$. Since the Dirichlet eigenvalues are monotone in D , λ_j is bounded from above by the j th eigenvalue of this square. The eigenvalues for this square are given by

$$\lambda_{k,l} = \pi^2(k^2 + l^2)/(2R^2), \quad k \in \mathbb{Z}^+, l \in \mathbb{Z}^+. \quad (5.5)$$

By definition

$$j = \#\{(k, l) : k^2 + l^2 \leq 2\lambda_j R^2/\pi^2\}. \quad (5.6)$$

Suppose $j \geq 4$. Then $k^2 + l^2 \geq 8$ since $j = 1$ corresponds to $(k, l) = (1, 1)$ and $j = 2, 3$ corresponds to $(k, l) = (2, 1)$ and $(k, l) = (1, 2)$. Hence

$$\lambda_j \geq \frac{4\pi^2}{R^2}, \quad j \geq 4. \quad (5.7)$$

But the right-hand side of (5.6) is equal to the number of lattice points in the first quadrant of the disc with radius $R(2\lambda_j/\pi^2)^{1/2}$. Hence by (5.6) and (5.7) we have for $j \geq 4$

$$j \geq \frac{\pi}{4} ((2\lambda_j R^2/\pi^2)^{1/2} - 2^{1/2})^2 \geq \frac{\lambda_j R^2}{8\pi}. \quad (5.8)$$

This proves the lemma for $j \geq 4$. The case $j = 1, 2, 3$ is easily verified. \square

Let F be the conformal map from the unit disc onto D with $F(0) = x$. Then by the results of [3, §1]

$$G_D(x, y) = \frac{1}{2\pi} \log \coth(\varrho_D(x, y)), \quad (5.9)$$

where

$$\varrho_D(x, y) = \inf_{\gamma} \int_0^1 \frac{|\gamma'(t)|}{|F'(0)|} dt, \quad (5.10)$$

and where the infimum is taken over all rectifiable curves γ in D with $\gamma(0) = x$, $\gamma(1) = y$, and where $F'(0)$ is evaluated at $\gamma(t)$. By Koebe's $1/4$ theorem

$$d(\gamma(t)) \leq |F'(0)| \leq 4d(\gamma(t)). \quad (5.11)$$

Without loss of generality we may assume that γ has a parametrisation with constant speed c . Then for any such γ we have

$$d(\gamma(t)) \leq d(x) + tc. \quad (5.12)$$

By (5.10)–(5.12)

$$\varrho_D(x, y) \geq \frac{1}{4} \int_0^1 \frac{c}{d(x) + tc} dt = \frac{1}{4} \log \left(1 + \frac{c}{d(x)} \right). \quad (5.13)$$

Since $c \geq |x - y|$ we have by (5.9) and (5.13)

$$G_D(x, y) \leq \frac{1}{2\pi} \log \frac{(d(x) + |x - y|)^{1/2} + d(x)^{1/2}}{(d(x) + |x - y|)^{1/2} - d(x)^{1/2}}. \quad (5.14)$$

We note that the right-hand side of (5.14) is positive and strictly decreasing in $|x - y|$ for x fixed. Hence the square of the right-hand side of (5.14) is strictly decreasing in $|x - y|$ for x fixed. Let R_0 be defined by

$$\pi R_0^2 = |D|. \quad (5.15)$$

By spherical-symmetric rearrangement

$$\begin{aligned} \int_D G_D^2(x, y) dy &\leq \frac{1}{2\pi} \int_0^{R_0} r dr \left(\log \frac{(d(x) + r)^{1/2} + d(x)^{1/2}}{(d(x) + r)^{1/2} - d(x)^{1/2}} \right)^2 \\ &= \frac{2d(x)^2}{\pi} \int_{d(x)/R_0}^{\infty} \frac{dr}{r^3} (\log((1 + r)^{1/2} + r^{1/2}))^2 \leq \frac{8d(x) R_0}{\pi}, \end{aligned} \quad (5.16)$$

since $\log((1 + r)^{1/2} + r^{1/2}) \leq 2r^{1/2}$. The theorem follows from (5.3), (5.4) and (5.16). \square

COROLLARY 5.2. *Let D be open, simply connected in \mathbb{R}^2 with finite volume $|D|$. Then*

$$\mathbb{E}_x[T_D] \leq 2^{3/2} \pi^{-3/4} |D|^{3/4} d(x)^{1/2}. \quad (5.17)$$

Proof. By the Cauchy–Schwarz inequality

$$\mathbb{E}_x[T_D] = \int_D G_D(x, y) dy \leq |D|^{1/2} \left\{ \int_D G_D^2(x, y) dy \right\}^{1/2} \quad (5.18)$$

and (5.17) follows from (5.18), (5.16) and (5.15). \square

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*School of Mathematics
University of Bristol
University Walk
Bristol BS8 1TW*

*Institut für Angewandte Mathematik
Universität Zürich
Winterthurer Straße 190
CH-8057 Zürich
Switzerland*

M.vandenBerg@bristol.ac.uk

eb@amath.unizh.ch